Cubic Splines (continued).

The equations for the second derivatives are given by

$$h_{i-1}S_{i-1} + (2h_{i-1} + 2h_i)S_i + h_i S_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) \qquad i=2,\dots,n-1 \qquad (14)$$

We choose the natural cubic splines by setting S_i =0, i=1,n. This is a tridiagonal linear system of equations. The matrix system is of the form

$$\begin{bmatrix} a_1 & b_2 & & & 0 \\ b_2 & a_2 & b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-3} & a_{n-3} & b_{n-2} \\ 0 & & & b_{n-2} & a_{n-2} \end{bmatrix} \begin{bmatrix} S_2 \\ S_3 \\ \vdots \\ S_{n-2} \\ S_{n-1} \end{bmatrix} = \text{RHS}. \qquad (15)$$

Where

$$\begin{aligned} a_i &= 2(h_i + h_{i+1}) \\ b_i &= h_i \end{aligned} \qquad (16)$$

This system can be solved using Gaussian elimination, or LU decomposition.

Example.

Fit the data in the Table by a cubic spline curve. (The true relation is just $y = x^3 - 8$.

X	Y
0	-8
1	-7
2	0
3	19
4	56

The linear system for the natural cubic spline is

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} S_2 \\ S_3 \\ S_4 \end{bmatrix} = \begin{bmatrix} 36 \\ 72 \\ 108 \end{bmatrix} \qquad (17)$$

This system can be solved by hand or directly using MATLAB. Set up the matrix and the RHS vector. Then the solution is given by $A^{-1}. b$, where A is the matrix and b is the RHS vector.

The solution is

S1=0, S2 = 6.4285, S3 = 10.2857, S4= 24.4285, S5=0.

We can then find the coefficients a_i, b_i, c_i, d_i by substitution.

Exercise. Find the natural cubic Splines for the following data:

	x_1	x_2	x_3
x	-1	0	1
y	1	0	1

The solution is:

$$y(x) = \frac{x^3}{2} + \frac{3}{2}x^2 \qquad \text{x in } [-1,0] \qquad (18.1)$$

$$y(x) = -\frac{x^3}{2} + \frac{3}{2}x^2 \qquad \text{x in } [0,1] \qquad (18.2)$$

Solve this problem and plot it using MATLAB.
The MATLAB function is called spline and its usage is as follows:

`yi = spline(x,y,xi)` where
x = [x1 x2 … xn] and y = [y1 y2 … yn] are the given vectors of (x_i, y_i) data points.
xi is the set of points where the spline is to be evaluated, and
yi are the corresponding values of the spline at xi.

Does the MATLAB solution correspond to the solution given by (18)?
Hint. Look at the values of the functions at the points 0, 0.1,0.2,0.3,…, 1