Polynomial Interpolation

Given values of an unknown function corresponding to certain values of x, i.e. (x_i,y_i), what is the behaviour of the function. We would like to answer the question "what is the function?" but this is impossible to tell with a limited amount of data.

The purpose in determining the behaviour of the function, as specified by the sample of data pairs (x,f(x)), is several-fold. We may wish to approximate other values of the function at values of x not tabulated (interpolation or extrapolation), and the integral of f(x) and its derivative. These objectives will lead us into ways of solving ordinary (ODEs) and partial differential equations (PDEs).

The strategy we will use in approximating unknown values of the function is straightforward. We will find a polynomial that fits a selected set of points $(x_i,f(x_i))$, and assume that the polynomial and the function behave nearly the same over the interval in question. Values of the polynomial should then be reasonable estimates of the values of the unknown function. When the polynomial is of the first degree, this leads to the familiar linear interpolation. We will be interested in polys. of higher degree, so that we can approximate functions that are far from linear.

DIFFERENCE TABLES.

We consider the following table of values of x and f(x). Each of the columns to the right of the f(x) column is computed by calculating the difference between two values in the column to its left.

x	f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)
0.0	0.000				
		0.203			
0.2	0.203		0.017		
		0.220		0.024	
0.4	0.423		0.041		0.020
		0.261		0.044	
0.6	0.684		0.085		0.052
		0.346		0.096	
0.8	1.030		0.181		0.211
		0.527		0.307	
1.0	1.557		0.488		
		1.015			
1.2	2.572				

Symbols that represent the entries in a difference table will be useful in using the table of differences to determine coefficients in interpolating polynomials. We use the letter h to stand for the uniform difference in the x-values, $h = \Delta x$. Using subscripts to represent the order of the x and f(x) values, we define the first differences of the function as

$$\Delta f_1 = f_2 - f_1, \qquad \Delta f_2 = f_3 - f_2, \qquad \Delta f_i = f_{i+1} - f_i. \qquad (1)$$

These are called forward differences. The second and higher order differences are similarly defined:

$$\Delta^2 f_1 = \Delta(\Delta f_1) = \Delta(f_2 - f_1) = \Delta f_2 - \Delta f_1 = (f_3 - f_2) - (f_2 - f_1)$$
$$= f_3 - 2f_2 + f_1$$

$$\Delta^2 f_i = f_{i+2} - 2f_{i+1} + f_i$$

$$\Delta^3 f_1 = \Delta(\Delta^2 f_1) = f_4 - 3f_3 + 3f_2 - f_1$$

$$\Delta^3 f_i = \Delta(\Delta^2 f_i) = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i$$

$$\Delta^n f_i = f_{i+n} - nf_{i+n-1} + \frac{n(n-1)}{2!} f_{i+n-2} - \frac{n(n-1)(n-2)}{3!} f_{i+n-3} + \ldots + (-1)^n f_i \qquad (2)$$

The pattern of coefficients in eqns. (2) is the familiar array of coefficients in the binomial expansion. 2nd and higher-order differences are usually obtained from the previous differences.

When f(x) behaves like a polynomial for the set of data points, the difference table has special properties. In the next table a function is tabulated over the domain x=0 to x=6, and f(x) obviously behaves the same as x^3. However we only know the values of f(x) in the Table.

x	f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)
0	0				
		1			
1	1		6		
		7		6	
2	8		12		0
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

We note that the 3rd differences are constant. Hence the 4th and all higher differences will be zero. In fact, the nth-order differences of any nth-degree polynomial are constant. This is analogous to differentiation of polynomials.

Interpolating polynomials.

When a function that is tabulated behaves like a polynomial (by observing that its nth-order differences are constant or nearly so),we can approximate it by the polynomial that it resembles. Our problem is to find the simplest means of writing the nth-degree polynomial that passes through n+1 pairs of points (x_i, f_i), i = 0,1,…n.
Note that such a polynomial is ***unique,*** there is only one poly of degree n passing through n+1 points.

One of the easiest ways to write a polynomial that passes through a group of **equispaced** points is the Newton-Gregory forward polynomial:

$$P_n(s) = P_n(x_s) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \dots$$
$$= f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + \binom{s}{3}\Delta^3 f_0 + \binom{s}{4}\Delta^4 f_0 + ... \qquad (3)$$

The variable s denotes the subscript on x and serves to index the x-values. $\binom{s}{n}$ denotes the number of combinations of s things taken n at a time. We can arbitrarily choose the origin for the subscripted variable, because by the use of negative and zero values of s we can refer to x-entries that precede x_1. We can now observe that $P_n(x)$ does match the table at all the data pairs $(x_i, f_i), i = 0,1,\dots,n.$

When s=0, $P_n(x_0) = f_0$.
If s=1, $P_n(x_1) = f_0 + \Delta f_0 = f_0 + f_1 - f_0 = f_1$.
If s=2, $P_n(x_2) = f_0 + 2\Delta f_0 + \Delta^2 f_0 = f_0 + 2(f_1 - f_0) + (f_2 - 2f_1 + f_0) = f_2$.

Similarly we can show that $P_n(x)$ formed by eqn. (3) matches at all n+1 points.
If over the domain from x_0 to x_n, $P_n(x)$ and f(x) have the same values at x_i, it is reasonable to assume that they will be nearly the same at intermediate x-values. However some error is to be expected in the estimate from such interpolation. We use the polynomial in (3) as an interpolation polynomial by letting s take on non-integral values, i.e. not integers. Note that for any value of x

$$s = \frac{x - x_0}{h} \quad \text{or} \quad x = x_0 + sh\,.$$

Exercise: Find the interpolation polynomial of degree 2 for the data points (0,1), (2,1), (4, 9).