LU Decomposition. (continued).

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & -4/7 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 7/3 & 7/3 \\ 0 & 0 & -1 \end{bmatrix}$$

For numerical stability we try to make the multipliers < 1. This means that we may have to interchange rows when carrying out the elimination. We can keep track of the interchanges by using permutation matrices P. Each row of a permutation matrix has exactly one 1 on each row and zeros everywhere else. E.g. to interchange rows i and j we put the (i,j) element =1, and (j,i) element =1, and all other rows have 1 on the main diagonal. Then the LU decomposition gives

$$LU = PA \tag{1}$$

where P is a product of permutation matrices, i.e another permutation matrix.
In the Example in the MATLAB handout for lu(X) we have from (1)

$$(P^{-1}L)U = P^{-1}PA = A \tag{2}$$

and $P^{-1}L$ is a permutation of the rows of L. The inverse of a permutation matrix is just the transpose of the matrix itself, i.e. a reflection in the main diagonal. In this example we compute PA to be

$$PA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \tag{3}$$

Verify the LU factorization of this matrix is as given with L lower triangular.
For this permutation you can check that all the multipliers are < 1.
The LU factorisation may also be written in the form

$$A = LDU \tag{4}$$

where D=$d_{i,i}$ is a diagonal matrix containing the elements of the main diagonal of U, and the ith row of U is divided by $d_{i,i}$, so that the main diagonal of U in (4) consists of only 1's.

Determinants.

The determinant of A is a scalar which determines if A is invertible (nonzero) or not. For a matrix of order 3, the determinant is given by the formula

$$\det(A) = \sum_{i,j,k=1}^{3} \varepsilon_{i,j,k} a_{1,i} a_{2,j} a_{3,k} \qquad i \neq j \neq k \neq i \tag{5}$$

where $\varepsilon_{i,j,k}$ is equal to 1 if (i,j,k) is an even permutation of (1,2,3), and is equal to -1 if it is an odd permutation of (1,2,3). Thus det(A) contains 6 terms. In general for a matrix of order n, det(A) contains n! terms. Thus for n = 10, the determinant using (5) contains 3,628,800 terms. The point here is that this formula (which is the theoretical one obtained by expanding the co-factors) is not very practical.

However det(*) satisfies the following rule det(AB)=det(A)det(B), so that we have

$$\det(A) = \det(LDU) = \det(L)\det(D)\det(U) = 1.\det(D).1 = \det(D) = d_{1,1}d_{2,2}\dots d_{n,n} \quad (6)$$

or if we use the LU form then the determinant is given by the product of the diagonal terms of the matrix U.
Verify the determinant of the examples done using the above product formula and using the det(A) MATLAB function.

If A is a symmetric **positive definite matrix** , then there exists a symmetric decomposition of A called the Cholesky decomposition such that A is given by

$$A = LL^T \quad (7)$$

where L is a lower triangular matrix with positive diagonal elements. A is a positive definite matrix if (x,Ax) > 0 for all vectors x <> 0.(x,Ax) denotes the inner-product (dot product) of x and Ax. If A is symmetric then the LDU decomposition can be written as LDL^T, i.e. L^T is the same as U.

Exercise. Find the Cholesky decomposition of the matrix

$$A = \begin{bmatrix} 10 & 7 & 8 \\ 7 & 5 & 6 \\ 8 & 6 & 10 \end{bmatrix} \qquad L = \begin{bmatrix} l_{1,1} & 0 & 0 \\ l_{2,1} & l_{2,2} & 0 \\ l_{3,1} & l_{3,2} & l_{3,3} \end{bmatrix} \quad (8)$$

Positive definite matrices arise in the discretisation of finite difference and finite element methods of partial differential equations. A symmetric matrix A is positive definite if $x^T Ax > 0$ for any non-zero vector x.
A is positive definite if its determinant, and the determinants of all its principal minors are greater then zero. Show this for the above matrix.

MATRIX AND VECTOR NORMS.
We need some way to measure the magnitude of matrices and vectors. For ordinary scalars this is just the absolute value. For vectors and matrices this is called the **norm**.
Any norm must satisfy certain conditions, we use $\|A\|$ to represent the norm of a matrix.

1. $\|A\| \geq 0$, and $\|A\| = 0$ if and only if A = 0.
2. $\|kA\| = k\|A\|$
3. $\|A+B\| \leq \|A\| + \|B\|$ Triangle inequality.
4. $\|AB\| \leq \|A\|\|B\|$ Cauchy-Schwarz inequality.

We define our norms to be consistent with the more familiar vector norms.
Some of these are :

1. $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ = sum of magnitudes
2. $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$ = Euclidean norm
3. $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ = maximum (infinity) norm.

Which of these norms is best to use depends on the problem.

Example. Compute the 1-, 2- and ∞-norms of the vector x=(1.25, 0.02,-5.15,0). The norms of a matrix are developed by a correspondence to vector norms. Matrix norms corresponding to the above, for matrix A, can be shown to be

1. $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{i,j}|$ = maximum column-sum
2. $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{i,j}|$ = maximum row-sum

The matrix norm $\|A\|_2$ that corresponds to the 2-norm of a vector is not readily computed. It is related to the eigenvalues of the matrix. It sometimes has a special use because no other norm is smaller than this norm. It is the maximum of the magnitudes of the eigenvalues. It thus provides the tightest measure of the size of a matrix, but is the most difficult to compute. This norm is also called the **spectral** norm.

The condition number of a matrix is defined to be

$$cond(A) = \|A\|_2 \|A^{-1}\|_2 \quad \text{or} \quad cond\,(A) = \|A\|_k \|A^{-1}\|_k \quad k=1,2,\ldots,\infty \tag{9}$$

If this number is large then the linear system with matrix A is said to be ill-conditioned. Ill-conditioned systems are difficult to solve accurately. This means that a small change in the RHS vector causes a large change in the solution vector x. This happens if A is *nearly* singular.

In MATLAB the function **cond(A)** computes the condition number of A. A (in)famous well-known ill-conditioned matrix is the Hilbert matrix of order n. For n=3 it is

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \tag{10}$$

EXERCISE. Using MATLAB find LU decomposition, A^{-1}, det(A), det(A^{-1}) using the L and U factors, and the condition number of A. See also the function **norm** for computing various norms of a matrix A.

EXERCISE. Find the LU decomposition, inverse and condition numbers of the Hilbert matrices of order n = 4, 5.

A very useful and informative web site on matrices among other things is at
http://www.sosmath.com/matrix/matrix.html

An interesting article on MATLAB and matrices can be found at

http://www.mathworks.com/company/newsletters/digest/sept01/