First Order Hermite Interpolation.

Frequently we may want to interpolate not only the values of a function at the points x_i, but also the values of the derivatives (slopes) at the points x_i, i =1,…,n as well. So we wish to find a polynomial p(x) such that,

$$P(x_i) = f_{i,} \quad \text{and} \quad P'(x_i) = f'_{i,} \qquad i = 1,\dots,n \tag{1}$$

So we have 2n conditions to satisfy. We can do this with a polynomial of degree 2n-1. *Hermite's formula*, exhibits a polynomial of degree 2n-1 which interpolates both the values of f and its first derivative at the points x_i. It has the form

$$P(x) = \sum_{i=1}^{n} U_i(x) f_i + \sum_{i=1}^{n} V_i(x) f_i' \tag{2}$$

The functions $U_i(x)$ and $V_i(x)$ are polynomials having properties similar to those of the Lagrange multipliers $l_i(x)$ presented earlier. In fact,

$$U_i(x) = [1 - 2(x - x_i) l_i'(x_i)][l_i(x)]^2 \quad \text{and} \tag{3a}$$

$$V_i(x) = (x - x_i)[l_i(x)]^2 \tag{3b}$$

The polynomials $U_i(x)$ and $V_i(x)$ satisfy the following conditions:

$$U_i(x_k) = \delta_{i,k,} \qquad U'_i(x_k) = 0 \tag{4a}$$

$$V_i(x_k) = 0 \qquad V'_i(x_k) = \delta_{i,k} \qquad \text{by construction.} \tag{4b}$$

Clearly also $U_i(x)$ and $V_i(x)$ are polynomials of degree 2n-1, and satisfy equation (1).

$$P(x_j) = \sum_{i=1}^{n} U_i(x_j) f_i + \sum_{i=1}^{n} V_i(x_j) f_i' = \sum_{i=1}^{n} \delta_{i,j} f_i + \sum_{i=1}^{n} 0 f'_i = f_j$$

$$P'(x_j) = \sum_{i=1}^{n} U'_i(x_j) f_i + \sum_{i=1}^{n} V'_i(x_j) f_i' = \sum_{i=1}^{n} 0 f_i + \sum_{i=1}^{n} \delta_{i,j} f'_i = f'_j$$

So for 2 points we get cubic polynomials for $U_i(x)$ and $V_i(x)$.
Example n=2, i.e. nodes {0,1}, show that

$$U_1(x) = (1 + 2x)(1 - x)^2 \quad \text{and} \quad U_2(x) = (3 - 2x)(x)^2 \tag{5}$$

$$V_1(x) = (x)(1 - x)^2 \quad \text{and} \quad V_2(x) = (x)^2(x - 1) \tag{6}$$

Cubic Spline Interpolation.

One technique that is becoming increasingly important is the so-called *spline fitting* of a curve. The conditions for a cubic spline are that we pass a set of cubics through the points, using a new cubic in each interval. To correspond to the idea of the draughtsman's spline, we require that both the slope and the curvature be the same for

the pair of cubics that join at each point. We now develop the equations subject to these conditions.

We write the cubic for the *i*th interval, which lies between the points (x_i,y_i) and (x_{i+1},y_{i+1}) in the form:

$$y_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \tag{7}$$

Since it fits at the two endpoints of the interval:

$$y_i = y_i(x_i) = a_i(x_i-x_i)^3 + b_i(x_i-x_i)^2 + c_i(x_i-x_i) + d_i = d_i \tag{8}$$

$$\begin{aligned} y_{i+1} = y_i(x_{i+1}) &= a_i(x_{i+1}-x_i)^3 + b_i(x_{i+1}-x_i)^2 + c_i(x_{i+1}-x_i) + d_i \\ &= a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i \end{aligned} \tag{9}$$

In the last equation we use $h_i = x_{i+1} - x_i$, for Δx in the ith interval. We need the first and second derivatives to relate the slopes and curvatures of the joining polynomials

$$y_i'(x) = 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i \tag{10}$$

$$y_i''(x) = 6a_i(x-x_i) + 2b_i \tag{11}$$

The mathematical procedure is simplified if we write the equations in terms of the second derivatives of the interpolating cubics. Let $S_i = y''(x_i)$ represent the second derivative at the point (x_i,y_i) and S_{i+1} at the point (x_{i+1},y_{i+1}). From (11) we have

$$\begin{aligned} S_i &= 6a_i(x_i-x_i) + 2b_i \\ S_i &= 2b_i \\ S_{i+1} &= 6a_i(x_{i+1}-x_i) + 2b_i \\ &= 6a_i h_i + 2b_i \end{aligned}$$

Hence we can write

$$b_i = S_i/2, \tag{12}$$

$$a_i = (S_{i+1} - S_i)/6h_i, \tag{13}$$

We substitute the relations for a_i, b_i, d_i given by eqns. (8), (12), (13) into equation (9) and then solve for c_i:

$$y_{i+1} = \left(\frac{S_{i+1}-S_i}{6h_i}\right)h_i^3 + \frac{S_i}{2}h_i^2 + c_i h_i + y_i$$

$$c_i = \frac{y_{i+1}-y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6} \tag{14}$$

We now invoke the condition that the slopes of the two cubics that join at (x_i,y_i) are the same. For the equation in the ith interval, eqn. (10) becomes, with $x=x_i$,

$$y_i'(x_i) = 3a_i(x_i-x_i)^2 + 2b_i(x_i-x_i) + c_i = c_i$$

In the previous interval, from (x_{i-1},x_i), the slope at its right end will be

$$\begin{aligned} y_{i-1}'(x_i) &= 3a_{i-1}(x_i-x_{i-1})^2 + 2b_{i-1}(x_i-x_{i-1}) + c_{i-1} \\ &= 3a_{i-1}h_{i-1}^2 + 2b_{i-1}h_{i-1} + c_{i-1} \end{aligned}$$

Equating these and substituting for a, b, c, d their relationships in terms of S and y, we get

$$y_i' = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6}$$
$$= 3(\frac{S_i - S_{i-1}}{6h_{i-1}})h_{i-1}{}^2 + 2(\frac{S_{i-1}}{2})h_{i-1} + \frac{y_i - y_{i-1}}{h_{i-1}} - \frac{2h_{i-1}S_{i-1} + h_{i-1}S_i}{6}$$

On simplifying this equation we get

$$h_{i-1}S_{i-1} + (2h_{i-1} + 2h_i)S_i + h_i S_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) \quad \text{i=2,…,n-1} \qquad (15)$$

Equation (15) applies at each internal point, from i=2 to i=n-1; n being the total number of points. This gives n-2 equations relating the n values of S_i. We get two additional equations involving S_1 and S_n, when we specify conditions pertaining to the end intervals of the whole curve. To some extent these end conditions are arbitrary. Three possible choices are often used:

1. Take $S_1 = 0, S_n = 0$. This is called the *natural cubic spline* and is equivalent to assuming that the end cubics approach linearity at their extremities.
2. Take $S_1 = S_2, S_n = S_{n-1}$. This is equivalent to assuming that the end cubics approach parabolas at their extremities.
3. Take S_1 as a linear extrapolation from S_2 and S_3, and S_n as a linear extrapolation from S_{n-1} and S_{n-2}.

Relation 1 is called the natural spline, and it is frequently used. Equations (15) written in matrix form give n-2 equations , but n unknowns. We can eliminate two unknowns S_1 and S_n using choice 1 above. We solve this system for the unknown $S_i, i = 2, \ldots n$. After the S_i values are calculated , we get the values of a_i, b_i, c_i, d_i for each interval if we want to interpolate for new values of the function:

$$a_i = (S_{i+1} - S_i) / 6h_i,$$
$$b_i = S_i / 2,$$
$$c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6},$$
$$d_i = y_i$$