Partial Order (e.g. \( \leq \))

- A binary relation \( \rho \) on a set \( B \) is called a **partial order** on \( B \) if it is: reflexive, anti-symmetric, and transitive.
- \( \langle B, \rho \rangle \) is called a **partially ordered set** or **poset**.
- Example, the vertex set of a directed acyclic graph ordered by **reachability**. Reachability is a **partial order** \( \leq \) on vertices, where \( u \leq v \) exactly when there exists a directed path from \( u \) to \( v \).
- A **total or linear order** is a partial order such that for all \( a, b \in A \), either \( a \rho b \) or \( b \rho a \).
Hasse diagrams

- Hasse diagrams represent partial orders (reflexive, anti-symmetric, transitive). When reading there is an implied upward orientation e.g. lower < upper. A point is drawn for each element of the poset, and line segments representing relations are drawn between these points according to the following two rules:
  - 1. If $x \leq y$ in the poset, then the point corresponding to $x$ appears lower in the drawing than the point corresponding to $y$.
  - 2. The line segment between the points corresponding to any two elements $x$ and $y$ of the poset is included in the drawing iff $x$ covers $y$ or $y$ covers $x$. 
Hasse diagrams

• The implicit relations in a Hasse diagram are reflexive and transitive

• The explicit relations in a Hasse diagram is anti-symmetry.
Hasse diagrams

- An element $z$ of a partially ordered set $(X,\leq)$ covers another element $x$ provided that there exists no third element $y$ in the poset for which $x \leq y \leq z$. In that case, $z$ is called an upper cover of $x$ and $x$ a lower cover of $z$. 

The covering relation is $\{(1,2), (1,3), (1,5), (1,7), (2,4), (2,6), (3,6), (4,8)\}$.
Partial Order for divides $|$ and $<$

**Hasse Diagrams for Finite Posets**

\[
\text{poset } \langle 1..9, | \rangle \quad \text{poset } \langle 0..3, < \rangle
\]
Divides by Relation

poset (\{1,2,3,4,6,8,12\}, |)
Order Relation on Power set\(^1\).

- A partially order set can be represented with using a POSET diagram. The POSET diagram on the right is based on the power set (all possible subsets) of the three element set \{a, b, c\}. These subsets form a special kind of partial order that is referred to as a lattice.
Hasse diagrams

• The Hasse diagram below represents information on a set of college computing courses and their prerequisites. The prerequisites form a partial order.
• Relating the prerequisites (partial order) to the diagram every course is dependent on Comp101,
• Comp252 covers Comp250, but not Comp201
• Comp341 directly depends on Comp251 and Comp252
Example: Constructing a Hasse Diagram

- Table 1 (next slide) represents information on a set of college courses and their prerequisites. The prerequisites relation is a partial order. We also show a Hasse for the partial ordering of these courses.
Example: Constructing a Hasse Diagram

<table>
<thead>
<tr>
<th>Course</th>
<th>Prerequisites</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp. 101</td>
<td>none</td>
</tr>
<tr>
<td>Comp. 201</td>
<td>Comp. 101</td>
</tr>
<tr>
<td>Comp. 250</td>
<td>Comp. 101</td>
</tr>
<tr>
<td>Comp. 251</td>
<td>Comp. 250</td>
</tr>
<tr>
<td>Comp. 340</td>
<td>Comp. 201</td>
</tr>
<tr>
<td>Comp. 341</td>
<td>Comp. 340</td>
</tr>
<tr>
<td>Comp. 450</td>
<td>Comp. 201, 250</td>
</tr>
<tr>
<td>Comp. 500</td>
<td>Comp. 450, 251</td>
</tr>
</tbody>
</table>
Example: Constructing a Hasse Diagram
An relation on binary digits.

- Each source has one less ‘1’ digit than its target.
Order relation Integers related to relation on binary digits.

- How do binary digits relate to their values?
- What about the value relation ‘less than’ on integers?
All connected Posets on 4 elements
Ordered relation “divides by”

Let \( A = \{1,2,3,4,6,9,12,18,24\} \)

For two natural numbers \( m \) and \( n \) the divisibility relation (\( | \)) can be written \( n|m \) if \( n \) divides \( m \) without remainder. (reads “\( n \) divides \( m \).” e.g. 2 divides 4)
Ordered relation “partial order”

- Let $B = \{a, b, c, d, e\}$,
- Relation: $d \leq b$, $d \leq c$, $e \leq c$, $b \leq a$, $c \leq a$
- Transitivity and identity not shown.
Ordered relation “partition order”

- A partition of a positive integer $m$ is a set whose sum is $m$. A partition $P_1$ precedes a partition $P_2$ if the integers in $P_1$ can be added to obtain the partition $P_2$. Let $m=5$ then we have: $5, 3+2, 2+2+1, 1+1+1+1+1, 4+1, 3+1+1, 2+1+1+1$. 

```plaintext
\[
\begin{array}{cccc}
5 & 4+1 & 3+2 & 2+2+1 +1 \\
4+1 & 3+1+1 & 2+2+1+1 & P2 \\
3+2 & 3+1+1 & 2+2+1+1 & P1 \\
5 & 4+1 & 3+2 & 2+2+1 +1
\end{array}
\]
Ordered relation “partition order”

- Two element set \( \{p, q\} \).
Hasse diagram summary

- The subset relation \( \subseteq \) represent partial order (reflexive, anti-symmetric, transitive \( < \)). When reading there is an implied upward orientation e.g. lower \( < \) upper. A point is drawn for each element of the poset, and line segments representing relations are drawn between these points according to the following two rules:

  1. If \( x < y \) in the poset, then the point corresponding to \( x \) appears lower in the drawing than the point corresponding to \( y \).
  2. The line segment between the points corresponding to any two elements \( x \) and \( y \) of the poset is included in the drawing iff \( x \) covers \( y \) or \( y \) covers \( x \).

- Implicit relations reflexive and transitive
- Explicit relation anti-symmetry.
Example relations

• **Transitive**: \{1\} \subset \{1,2\} \subset \{1,2,3\}

• **Reflexive**: \{1\} \subseteq \{1\}

• **Cover**: A cover is the transitive reflexive reduction of a partial order. An element \(z\) (e.g. \{1,2\}) of a partially ordered set above \((X,\leq)\) covers another element \(x\) (e.g.\{1\} and \{2\}) provided that there exists no third element \(y\) in the poset for which \(x \leq y \leq z\).

• If we have \(x \leq y \leq z\), then \(z\) is called an upper cover of \(x\) and \(x\) a lower cover of \(z\).

• **Proper subsets of exactly one other set.**

• \{1,2\}, \{1,3\}, \{2,3\} \subset \{1,2,3\}
Minimal and Maximal Elements

- An element $a$ in $S$ is called a **minimal** element if no other element of $S$ strictly precedes $a$ (no edge enters $a$ from below).
- An element $b$ in $S$ is called a **maximal** element if no other element of $S$ strictly succeeds $b$ (no edge leaves $b$ from above).
- $S$ can have more than one maximal and more than one minimal element.
Maximal & Minimal examples

One minimal

Two maximal

One maximal

Two minimal
First and Last Elements

• An element $a$ in $S$ is a called first (or least) if $a \leq x$ for every element $x$ in $S$ (at bottom of page).

• An element $b$ in $S$ is a called last (or greatest) if $y \leq b$ for every element $y$ in $S$ (at top of page).

• $S$ may have neither a first or a last element.

• $S$ can have at most one first element, which must be minimal.

• $S$ can have at most one last element, which must be maximal.
First & Last examples

One minimal, which is also first

Two maximal, neither is a last

---

One maximal, which is also last

Two minimal, neither a first.
First, Last, Maximal, Minimal

- Hasse diagram on left is ordered by set inclusion.
- $U =$ Last and Maximal.
- $\emptyset =$ First and Minimal
Partial order

• Let \( D = \{1,2,4,5,10,20,25,50,100\} \)
• \( \forall x,y \in D \) let \( d(x,y) \) (meaning \( x \) divides \( y \) evenly) form partial order \( x \leq y \).
• Let \( S = \{10,20,50\} \) where \( S \subseteq D \)

• Find each of the following:
  • The minimal elements of \( S \).
  • The maximal elements of \( S \).
  • The lower bond of \( S \).
  • The upper bound of \( S \).
  • The GLB of \( S \) (not covered yet)
  • The LUB of \( S \) (not covered yet)
Sub set of a partial order

Let $D = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$

Let $S = \{10, 20, 50\}$ where $S \subseteq D$
Recall properties of relations

- **Reflexive** \( (\forall b \mid b \rho b) \)
- **Irreflexive** \( (\forall b \mid \neg (b \rho b)) \)
- **Symmetric** \( (\forall b, c \mid (b \rho c) \equiv (c \rho b)) \)
- **Antisymmetric**
  \( (\forall b, c \mid (b \rho c) \land (c \rho b) \Rightarrow b = c) \)
- **Asymmetric** (non-symmetric, see notes section)
  \( (\forall b, c \mid (b \rho c) \Rightarrow \neg (c \rho b)) \)
- **Transitive**
  \( (\forall b, c, d \mid (b \rho c) \land (c \rho d) \Rightarrow b \rho d) \)
Sensible Closures

• Reflexive Closure $\mathcal{r}(\rho)$
  \[- (\rho \cup \rho^0)\]

• Symmetric $\mathcal{s}(\rho)$
  \[- (\rho \cup \rho^{-1})\]

• Other closures include Transitive closure $\rho^+$, Reflexive Transitive closure $\rho^*$
Sample closures

• Let $R = \{ <a, b>, <c, a>, <c, c> \}$ be a relation on the set $A = \{ a, b, c \}$.

• The reflexive closure is:
  
  $r(R) = \{ <a, b>, <c, a>, <a, a>, <b, b>, <c, c> \}$

• The symmetric closure is:
  
  $s(R) = \{ <a, b>, <c, a>, <c, c>, <b, a>, <a, c> \}$
Equivalence Relations

- A relation is an equivalence relation iff it is reflexive, symmetric and transitive (e.g. =).
- An equivalence relation $\rho$ on a set $B$ partitions the set into non-empty disjoint subsets. Elements that are equivalent under $\rho$ are placed in the same partition. Elements that are not equivalent under $\rho$ are placed in different partitions. For example:

  $\forall \langle b, c \rangle \in \text{sameEye} \equiv b$ and $c$ have same eye colour
Partial Order, Linear Order

• A binary relation $\rho$ on a set $B$ is called a **partial order** on $B$ if it is reflexive, antisymmetric, and transitive. $<B, \rho>$ is called a **partially ordered set** or **poset**.

• A linear order is a partial order such that for all $a, b \in A$, either $a \rho b$ or $b \rho a$. 
Partial Order, Notation

- Finite partially ordered sets can be represented in a diagram where elements of the set are represented by nodes and a line connecting two nodes indicates that the lower of them is related to the upper. Reflexivity and Transitivity are assumed but not shown.
Whole to part Order

Whole to part relation
Everything is a component of A. What about J?
<table>
<thead>
<tr>
<th>Courses Needed</th>
<th>Prerequisites</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculus I</td>
<td></td>
</tr>
<tr>
<td>Calculus II</td>
<td>Calculus I</td>
</tr>
<tr>
<td>Calculus III</td>
<td>Calculus II</td>
</tr>
<tr>
<td>Linear Algebra</td>
<td>Calculus II</td>
</tr>
<tr>
<td>Programming II</td>
<td>Calculus I</td>
</tr>
<tr>
<td>Software Development</td>
<td>Programming II</td>
</tr>
<tr>
<td>3D Computer Graphics</td>
<td>Calculus III, Linear Algebra, Programming II</td>
</tr>
</tbody>
</table>
Use of lattices

• A lattice is a partially ordered set in which every two elements have a least upper bound and a greatest lower bound.

• An example is given by the natural numbers, partially ordered by divisibility, for which the least upper bound is the least common multiple and the greatest lower bound is the greatest common divisor.

• $\text{GCD}(8,12) = 4$

• $\text{LCM}(4,6) = 12$
Use of lattices

• Lattices can be used for knowledge representation, such as Formal Concept Analysis (FCA), semantic web, class hierarchies (check the web).

• Lattices are present in specification and programming languages e.g. relating CafeOBJ sorts, CafeOBJ module imports form a partial order\(^1\).

• Assertions about programs have a special relation with each other and form a *lattice structure*. 
Specification

• Mathematics is an appropriate linguistic tool for expressing specifications of algebras, lattices, sets, graphs.

• Lattices are important both as examples of a kind of algebra, and also used in the study of other kinds of algebra.

• Each algebra has an associated lattice.
Lattice: preliminary definitions

• Given a partially ordered set \((A, R)\) and subset \(S \subseteq A\), then \(a \in A\) is a **lower bound** of \(S\) if: \(\forall x \in S . a \leq x\) (e.g. \(\forall x \in S . a \leq x\))

• Given a partially ordered set \((A, R)\) and subset \(S \subseteq A\), then \(b \in A\) is an **upper bound** of \(S\) if: \(\forall x \in S . x \leq b\) (e.g. \(\forall x \in S . x \leq b\))

• \(A, S\) denote sets and \(S\) is smaller than or equal to \(A\). Also, \(a\) and \(b\) are not necessarily in \(S\), they could be in \(A\) (bigger set).
What are the upper and lower bounds of $S$?
$S = \{a, b, c\}$

- Upper bound of $\{a, b, c\}$
- Lower bound of $\{a, b, c\}$

Subset $S$
Lattice: preliminary definitions

- $a$ is called the \textcolor{red}{greatest lower bound} (glb) of a set $S$ if $a$ is the greatest of all lower bounds.
- $a \in S \land \forall l \in A. l \in S \Rightarrow l \in R a$ (in general)
- $a \leq S \land \forall l \in A. l \leq S \Rightarrow l \leq a$ (for example)
- We write $\cap S$ for glb of $S$. 
Lattice: preliminary definitions

- $b$ is called the least upper bound (lub) of $S$ if $b$ is the least of all upper bounds.

- $S \leq b \land \forall u \in A. S \leq b \Rightarrow b \leq u$ (in general)

- $S \leq b \land \forall u \in A. S \leq u \Rightarrow b \leq u$ (for example)

- We write $\sqcup S$ for lub of $S$. 
Lattice and Algebra

- \textit{LUB} is referred to as the Supremum (⊔S).
- \textit{GLB} is referred to as the Infimum (∩S).

\textbf{Definition 3.1.2.} If \((L, \leq)\) is a lattice then we define the following binary operations, named the\textit{ join} (or union or disjunction) and\textit{ meet} (or intersection or conjunction) operations respectively:

\[\vee : L \times L \rightarrow L, \ x \vee y = \sup\{x, y\}\]  
\[\wedge : L \times L \rightarrow L, \ x \wedge y = \inf\{x, y\}\]  

These operations are well defined because the least upper bound and the greatest lower bound are uniquely determined. We observe that they are "total" operations.
Top and Bottom elements

- For the lattice of implication ($\Rightarrow$)

\[
\begin{array}{c}
T \\
a \lor k \\
a \lor n \\
k \lor n \\
\downarrow \\
a \\
k \\
n
\end{array}
\]

\[
\begin{align*}
a &= \text{`Alice stole the tarts!'}; \\
k &= \text{`The Knave of Hearts stole the tarts!'}; \\
n &= \text{`No one stole the tarts!'};
\end{align*}
\]
This POSET A itself has no lub, but the subset S={a,b,c} has both a lub & glb.
Lattice: definition

- A partially ordered set in which every finite subset has a least upper bound and a greatest lower bound is called a lattice.
- A partially ordered set in which every subset (not just finite) has a lub and glb is called a complete lattice.
Subsets

Example: Let $P = \text{power} \{\{a, b, c\}\}$ shown in the picture and let $S = \{\{a\}, \{a, b\}, \{a, c\}\}$. Then $\{a\}$ is the only minimal element of $S$, so $\{a\}$ is the least element of $S$. The lower bounds of $S$ are $\{a\}$ and $\emptyset$, with $\text{glb}(S) = \{a\}$. The maximal elements of $S$ are $\{a, b\}$ and $\{a, c\}$, but there is no greatest element of $S$. The only upper bound of $S$ is $\{a, b, c\}$, so $\text{lub}(S) = \{a, b, c\}$.

An element $a$ in $S$ is a called **first** (or **least**) if $a \leq x$ for every element $x$ in $S$ (at bottom of page).

An element $b$ in $S$ is a called **last** (or **greatest**) if $y \leq b$ for every element $y$ in $S$ (at top of page).
Subsets

Find the minima, maxima, and bounds for the subset $S = \{6, 7, 8\}$ of the poset $P$ pictured poset diagram.

The minimal elements of $S$ are 6 and 7, but there is no least element of $S$. The lower bounds of $S$ are 1, 2, 4 with $\text{glb}(S) = 4$. The only maximal element of $S$ is 8, so 8 is also the greatest element of $S$. The upper bounds of $S$ are 8 and 9, with $\text{lub}(S) = 8$.

An element $a$ in $S$ is a called **first** (or **least**) if $a \leq x$ for every element $x$ in $S$ (at bottom of page).

An element $b$ in $S$ is a called **last** (or **greatest**) if $y \leq b$ for every element $y$ in $S$ (at top of page).
Partial Order Relation on Divisibility\(^1\).

- The set
  - \( A = \{ 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 \} \)
- of all divisors of 60, partially ordered by divisibility.
Partial Order Relation on Divisibility.

The set
\[ A = \{ 1, 2, 3, 5, 6, 10, 15 \} \]
contains all divisors of 30. The Hasse diagram partially ordered by divisibility.
Partial Order Relation.

- What is the set here?
- What is the relation?
- Is this a lattice?
Partial Order Relation.

• $S=\{\{1\},\{2\},\{3\},\{4\}, \{1,2\}, \{1,5\}, \{3,6\}, \{4,6\}, \{0,3,6\}, \{1,5,8\}, \{0,3,4,6\}\}$
• Don and Betty are Dave’s parents, Jack and Audrey are Amy’s parents. Dave and Amy are parents of John and Jess.
• $S=\{\text{John}, \text{Jess}, \text{Dave}, \text{Amy}, \text{Don}, \text{Betty}, \text{Audrey}, \text{Jack}\}$
Partial Order Relation.

• Let $S$ be a set of sets. Define $A \leq_R B$ to mean $A \subseteq B$.

• $R$ is an antisymmetric relation on $S$, because if $X \leq_R Y$ and $Y \leq_R X$ then:
  \[ X \subseteq Y \land Y \subseteq X \implies X = Y. \]

• $R$ is reflexive, because $Y \subseteq X$

• $R$ is transitive
Let \( D(n) \) denote the positive divisors of \( n \).

\[ L = D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\} \]

**Sub-lattices:** \( D(6), D(10), D(15), \{5, 10, 15, 30\} \)

For the divisor relation, if \( n \divides m \) then \( D(m) \) is a sub-lattice of \( D(n) \).
Hasse Diagram Example

• Given the relation \( \rho \) defined on the set \( A \)

• \( \rho = \{(x,y) \mid x \text{ is a factor of } y\} \)

• \( A = \{1, 2, 3, 4, 6, 10, 12, 20\} \)

• The next slide shows the Hasse diagram for the relation \( \rho \) on the set \( A \).
\( \rho = \{ (x, y) \mid x \text{ is a factor of } y \} \) \( A = \{ 1, 2, 3, 4, 6, 10, 12, 20 \} \). Is this a lattice?

No, because each subset does not have a lub or a glb. (e.g. \{20,12\} has no lub)
Subset of poset with lub & glb

Bounds may be in S. Recall that lattices are based on POSETs (reflexive, anti-symmetric, transitive), so

(glb) \( a \leq a, a \leq b, b \leq c, c \leq c \) (lub)

A subset may have a lower bound within itself or not within itself, and likewise for upper bounds.
This POSET A has no lub, but the subset $S=\{a, b, c\}$ has both a lub & glb.

**POSET A**

- **LUB of $\{a, b, c\}$**
- **GLB of $\{a, b, c\}$**
- **Subset $S$**
- **Nodes:** a, b, c, d, e, f, g, h, i, k, l
$S = \{ e, f, g, c \}$ has no LUB or any upper bounds, but has GLB.
LUB is the least of all upper bounds

GLB (if exists) is the greatest of all lower bounds
What is GLB of \( S = \{a, b, c\} \) ?

- Recall def. of GLB \( a \). (capital \( A, S \) are sets)
  
- \( a \leq S \land \forall l \in A. l \leq S \Rightarrow l \leq a \)
What is LUB of \( S = \{a, b, c\} \)?

- Recall def. of LUB \( b \). (capital A, S are sets)
- \( S \leq b \land \forall u \in A . S \leq u \implies b \leq u \)
LUB\( b : S \leq b \land \forall u \in A. S \leq u \Rightarrow b \leq u \)

\( b, \) if it exists, is the least of all upper bounds

h and i are lower bounds of \( S = \{a, b\} \), h is a lower bound of both, while i is a lower bound of b only.

So the set S has two lower bounds neither of which is greater. No GLB

GLB\( a : a \leq S \land \forall l \in A. l \leq S \Rightarrow l \leq a \)

a, if it exists, is the greatest of all lower bounds.
• **LUB** $b : S \leq b \land \forall u \in A. S \leq u \Rightarrow b \leq u$

• **GLB** $a : a \leq S \land \forall l \in A. l \leq S \Rightarrow l \leq a$
a poset, but not a lattice
Top and Bottom elements

- The element of a complete lattice which is the lub of the whole lattice is called top $\top$, and the glb is called bottom $\bot$.
- Sometimes the symbol $\sqsubseteq$ is used to represent a general relation e.g. for the lattice $(\mathbb{Z}, \leq)$
- $\sqsubseteq$ corresponds to $\leq$
- $\sqcap$ corresponds to Max (glb of two integers)
- $\sqcup$ corresponds to Min (lub)
Top and Bottom elements

- For the lattice $(\text{Bool, } \Rightarrow)$ (implies)
- $\sqsubseteq$ corresponds to $=$
- $\sqcap$ corresponds to $\land$ (glb)
- $\sqcup$ corresponds to $\lor$ (lub)
- Every two-element subset has a lub (supremum) and glb (infimum).
Example

- Which of the following partially ordered sets are lattices:

- A poset is a lattice iff for each pair \( x, y \) \( \text{lub}(x, y) \) and \( \text{glb}(x, y) \) both exist. On RHS \( \{a, b\} \) has three upper bounds \( c, d, \) and \( I \) and no one of them precedes the other two, i.e. none is least.
Power set\(^1\) (again)

- The POSET Lattice on the right is based on the power set of \(\{a, b, c\}\). The LUB is given by the union and the GLB by the intersection of subsets.
Sub Lattices

$L_1 = \{a, e, g, h\}$ is not a sub-lattice because $e \land g \not\in L_1$,
while $L_2 = \{a, c, e, h\}$ is.

In $L$ meet is $e \land g = c$
Formal Concepts Analysis is based on Lattice Theory.

- A branch of computing that uses lattices is called Formal Concept Analysis. FCA is based on the assumption that human knowledge involves conceptual thinking, and that human reasoning involves manipulation of concepts. FCA takes the view that a concept is a unit of thought constituted by its extension (values or instances) and its intension (seems intention is OK) (schemas or classes). These ideas go back over 2000 years to Aristotle.
Lattices can be used for Knowledge Representation

A useful knowledge representation for the semantic web. A line diagram consists of circles, lines and the names of all objects and all attributes of the given context. The circles represent the concepts and the information of the context can be read from the line diagram by the following simple reading rule: An object $g$ has an attribute $m$ if and only if there is an upwards leading path from the circle named by "$g$" to the circle named by "$m$".

<table>
<thead>
<tr>
<th>ANIMALS</th>
<th>preying</th>
<th>flying</th>
<th>bird</th>
<th>mammal</th>
</tr>
</thead>
<tbody>
<tr>
<td>LION</td>
<td>×</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>FINCH</td>
<td></td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>EAGLE</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>HARE</td>
<td></td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>OSTRICH</td>
<td></td>
<td></td>
<td></td>
<td>×</td>
</tr>
</tbody>
</table>
The top of the lattice contains all of the objects and none of the attributes, while the bottom of the lattice contains all of the attributes and none of the objects.
Formal Concepts Background

• “Adding axioms makes a theory larger, in the sense that more propositions become provable. But the larger theory is also more specialized, since it applies to a smaller range of possible models. This principle, which was first observed by Aristotle, is known as the inverse relationship between intension and extension: as the meaning or intension grows larger in terms of the number of axioms or defining conditions, the extension grows smaller in terms of the number of possible instances.” Sowa
Formal Concepts Background

• “As an example, more conditions are needed to define the type Dog than the type Animal; therefore, there are fewer instances of dogs in the world than there are animals. Even more axioms are needed to define the subtypes Dachshund or Collie, which have even fewer instances than the type Dog.”: Sowa
Formal Concepts Background

- A concept \((O,A)\) consists of Objects and Attributes.
- The **extension** of a concept \((O,A)\) is the collection of all objects \(O\) belonging to that concept.
- The **intension** of a concept \((O,A)\) is the collection of all attributes \(A\) belonging to that concept.
- Sub-concepts **satisfy** larger sets of axioms or attributes (usually less instances of them exist, \(\downarrow\)).
- Subsets of attributes **determine** super-concepts (usually more instances of them exist, \(\uparrow\)).
Formal Concepts Background

- There is a duality between objects and attributes called a Galois connection. A Galois connection implies that if one makes the set of objects larger, it corresponds to a smaller set of attributes, and vice versa.

- This particular Galois connection exhibits a closure of the relation between objects and attributes. From any set of formal objects one can identify all formal attributes which they have in common (and vice versa).
Formal Concepts Background

- The **top** and **bottom** concepts in a concept lattice are special.
- The top concept has all formal objects in its extension. Its intension is often empty but does not need to be empty. The top concept can be thought of as representing the “universal” concept of a formal context.
- The bottom concept has all formal attributes in its intension. The bottom concept the “null” or “contradictory” concept of a formal context.
Formal Concepts Background

• FCA is just a mathematic theory, like integers or sets. Caution is required when applying FCA to real world domains. Many formal concepts may correspond to intuitive notions, but not all formal concepts need to do so.

• FCA focuses on formal structure, it is the user’s responsibility to insure the formal context corresponds to some cognitive or real world entity (i.e. an idea or a thing). FCA not a formal analysis of human concepts, but instead is a mathematical method using formal concepts and contexts.
Definitions: Context & Concept

• Let $M$ be a set of attributes, $G$ be a set of objects, and $I$ a relation between $G$ and $M$

• $I$ is called the incidence relation of the formal context $K = (G,M,I)$

• A pair $(A,B)$ is said to be a formal concept of the formal context $(G,M,I)$ if $A \subseteq G$, $B \subseteq M$, $\sigma(A)=B$ and $\tau(B)=A$.

• $\mathcal{B}(G,M,I)$ denotes the set of all concepts in context $(G,M,I)$
Extent and Intent

• The actual OBJECTS A are the extent of the formal concept (A,B)
• The actual ATTRIBUTES B are the intent of the formal concept (A,B)
• Several objects may match the intent of a node exactly. They are said to be contingent. The size of the object contingent represents the number of objects for each concept.
Formal Concepts have an ordered relation

- Let $B(G,M,I)$ denote the set of all concepts of the context $(G,M,I)$. The concepts of a context are ordered by the subconcept-superconcept relation which is defined by:
  - $(A_1, B_1) \leq (A_2, B_2) \iff A_1 \subseteq A_2 \land B_2 \subseteq B_1$

- Which says:

- $(A_2,B_2)$ is a super-concept of $(A_1,B_1)$ or
- $(A_1,B_1)$ is a sub-concept of $(A_2,B_2)$

- Sub-concepts are said to be smaller or less general than their super-concepts and the super-concepts larger or more general than their sub-concepts.
Lab

Read “A first course in formal concept analysis” by Karl Erich Wolff
Lab

Download and install Concept Explorer from:

http://sourceforge.net/projects/conexp
http://conexp.sourceforge.net
What happens if we add a Bat?
The extent of a concept represents all the object labels that can be reached along a descending path from the concept. The set of concepts along the downward path is known as the *down-set* or *order ideal*. Conversely, the intent of a concept can be recovered by collecting all of the attribute labels along upward paths from the concept. The set of concepts along the upward paths are known as the *up-set* or *order filter*.

![Concept lattice diagram](image)

Figure 1.2: The formal concept lattice corresponding to the planet context in Table 1.1.

For example, the extent of the concept with the attribute label "small" in Figure 1.2 can be found by following the downward paths to recover the planets \{\textit{Venus, Mercury, Earth, Mars, Pluto}\}. That is, those planets which are small. The attributes or intent of the planet Pluto can be found by following the upward paths from the concept with the object label "Pluto" to recover the set \{\textit{small, moon(s), far}\}. 
The extent of a concept represents all the object labels that can be reached along a descending path from the concept. The set of concepts along the downward path is known as the down-set or order ideal.
Filter\textsuperscript{†}

• Conversely, the intent of a concept can be recovered by collecting all of the attribute labels along upward paths from the concept. The set of concepts along the upward paths are known as the up-set or order filter.
Retrieving Extension & Intension

• To retrieve the extension of a formal concept one needs to trace all paths which lead down from the node to collect the formal objects.

• To retrieve the intension of a formal concept one needs to trace all paths which lead up in order to collect all the formal attributes.
Example: all Object & All Attributes

• See 'planets.pdf' document on course web page.

• At the top of the lattice we have all the objects but no attributes (we know nothing about everything)

• At the bottom of the lattice we have no objects and all the attribute (we know everything about nothing)
Example 1

Reading Rule: An object g has an attribute m if and only if there is an upwards leading path from the circle named by "g" to the circle named by "m".
**Example 2**

<table>
<thead>
<tr>
<th>$\mathcal{T}$</th>
<th>four-legged</th>
<th>hair-covered</th>
<th>intelligent</th>
<th>marine</th>
<th>thumbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cats</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dogs</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dolphins</td>
<td></td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gibbons</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Humans</td>
<td></td>
<td>×</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Whales</td>
<td></td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mammal example: Table $\mathcal{T}$ represents the binary relations

<table>
<thead>
<tr>
<th>Concept</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>top</td>
<td>( ${ $Cats, Gibbons, Dogs, Dolphins, Humans, Whales $}$, $\emptyset$ )</td>
</tr>
<tr>
<td>$c_6$</td>
<td>${ $Gibbons, Dolphins, Humans, Whales $}$, ${ $intelligent $}$</td>
</tr>
<tr>
<td>$c_5$</td>
<td>${ $Cats, Gibbons, Dogs $}$, ${ $hair-covered $}$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>${ $Dolphins, Whales $}$, ${ $intelligent, marine $}$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>${ $Humans $}$, ${ $intelligent, thumbed $}$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>${ $Cats, Dogs $}$, ${ $hair-covered, four-legged $}$</td>
</tr>
<tr>
<td>$c_1$</td>
<td>${ $Gibbons $}$, ${ $hair-covered, intelligent, thumbed $}$</td>
</tr>
<tr>
<td>bottom</td>
<td>( $\emptyset$, ${ $four-legged, hair-covered, intelligent, marine, thumbed $}$ )</td>
</tr>
</tbody>
</table>

Concepts of the mammal example
Advantages of FCA

• FCA develops a mathematical theory of concepts, which consist of objects and attributes.
• FCA formally represent a Galois connection between ordered sets of objects and attributes.
• A Galois connection is a relation between two partially ordered sets (posets). In the FCA case the posets are objects and attributes.
• Sets of formal concepts can be visualized.
• Automated logical inference can be used.
Advantages of FCA

• FCA develops a mathematical and computable theory which can represent \textit{concepts}, which consist of objects and attributes.

• This mathematical theory can be \textit{visualized} in an intuitive way.

• Concept Analysis can be used to identify groupings of objects that possess common attributes.
Applications of FCA

• Constructing classification & taxonomies.
• Data mining
• Conceptual information systems
• Information retrieval systems
• Semantic Web
• Formally modelling OO class hierarchies
Intuitive approach to Constructing a Concept Lattice

<table>
<thead>
<tr>
<th>Concept</th>
<th>Attributes</th>
</tr>
</thead>
<tbody>
<tr>
<td>top</td>
<td>{ Cats, Gibbons, Dogs, Dolphins, Humans, Whales }, {}</td>
</tr>
<tr>
<td>$c_6$</td>
<td>{ Gibbons, Dolphins, Humans, Whales }, { intelligent }</td>
</tr>
<tr>
<td>$c_5$</td>
<td>{ Cats, Gibbons, Dogs }, { hair-covered }</td>
</tr>
<tr>
<td>$c_4$</td>
<td>{ Dolphins, Whales }, { intelligent, marine }</td>
</tr>
<tr>
<td>$c_3$</td>
<td>{ Humans }, { intelligent, thumbed }</td>
</tr>
<tr>
<td>$c_2$</td>
<td>{ Cats, Dogs }, { hair-covered, four-legged }</td>
</tr>
<tr>
<td>$c_1$</td>
<td>{ Gibbons }, { hair-covered, intelligent, thumbed }</td>
</tr>
<tr>
<td>bottom</td>
<td>{ }, { four-legged, hair-covered, intelligent, marine, thumbed }</td>
</tr>
</tbody>
</table>

Concepts of the mammal example
Intuitive approach to Constructing a Concept Lattice

1. Start at top with all objects and no attributes.

2. Make a concepts for the biggest set of attributes (i.e. intelligent and haircovered)

2.1 Are there any objects that exactly match these attributes? No, so we label the concepts with only the attributes.

   - ({Gibbons, Dolphins, Whales, Humans}, {intelligent}),
   - ({Gibbons, Dogs, Cats}, {haircovered}).
Intuitive approach to Constructing a Concept Lattice

3. Add one attribute at a time to the attribute sets.

- First marine and thumbed to intelligent
- Second four-legged to haircovered.

3.1 Are there any objects that exactly match these attributes? Yes, so we label the concepts with objects. Giving:

- \(\{\text{Dolphins, Whales}\}, \{\text{intelligent, marine}\}\)
- \(\{\text{Humans}\}, \{\text{intelligent, thumbed}\}\),
- \(\{\text{Cats, Dogs}\}, \{\text{four-legged, haircovered}\}\)
Intuitive approach to Constructing a Concept Lattice

3. This only leaves the Gibbon object. Are there any objects that exactly match these and previous attributes? Yes, giving new node:

- \(\{\text{Gibbon}, \{\text{intelligent, thumbed, haircovered}\}\}\)

Place these new nodes under the appropriate parent node, but we only label them with the current objects and current attributes (not with the inherited attributes)
Intuitive approach to Constructing a Concept Lattice

• Now all of the objects have be generated:
  • ({Dolphins, Whales}, {intelligent, marine})
  • ({Humans}, {intelligent, thumbed}),
  • ({Cats, Dogs}, {four-legged, haircovered})
  • ({Gibbon}, {intelligent, thumbed, haircovered})
Intuitive approach to Constructing a Concept Lattice

- 4. We have now individually covered all the objects, so we add the full collection to the bottom node.
- \((\emptyset, \{\text{intelligent, thumbed, four-legged, haircovered, marine}\})\)

<table>
<thead>
<tr>
<th>Concept</th>
<th>Attributes</th>
</tr>
</thead>
<tbody>
<tr>
<td>top</td>
<td>({\text{Cats, Gibbons, Dogs, Dolphins, Humans, Whales}}, \emptyset)</td>
</tr>
<tr>
<td>(c_6)</td>
<td>({\text{Gibbons, Dolphins, Humans, Whales}}, {\text{intelligent}})</td>
</tr>
<tr>
<td>(c_5)</td>
<td>({\text{Cats, Gibbons, Dogs}}, {\text{hair-covered}})</td>
</tr>
<tr>
<td>(c_4)</td>
<td>({\text{Dolphins, Whales}}, {\text{intelligent, marine}})</td>
</tr>
<tr>
<td>(c_3)</td>
<td>({\text{Humans}}, {\text{intelligent, thumbed}})</td>
</tr>
<tr>
<td>(c_2)</td>
<td>({\text{Cats, Dogs}}, {\text{hair-covered, four-legged}})</td>
</tr>
<tr>
<td>(c_1)</td>
<td>({\text{Gibbons}}, {\text{hair-covered, intelligent, thumbed}})</td>
</tr>
<tr>
<td>bottom</td>
<td>(\emptyset, {\text{four-legged, hair-covered, intelligent, marine, thumbed}})</td>
</tr>
</tbody>
</table>

Concepts of the mammal example
Age

<table>
<thead>
<tr>
<th>female</th>
<th>juvenile</th>
<th>adult</th>
<th>male</th>
</tr>
</thead>
<tbody>
<tr>
<td>girl</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>woman</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>boy</td>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>man</td>
<td></td>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>

Diagram:

- Female
- Juvenile
- Adult
- Male
- Girl
- Woman
- Boy
- Man

Diagram shows the relationships between different age and gender categories.
Live in water

<table>
<thead>
<tr>
<th>Needs water to live</th>
<th>Lives in water</th>
<th>Lives on land</th>
<th>Needs Chlorophyll</th>
<th>Monocotyledon</th>
<th>Can move</th>
<th>Has limbs</th>
<th>Breast feeds</th>
</tr>
</thead>
<tbody>
<tr>
<td>fish leech</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>bream</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>frog</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>dog</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>water weeds</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
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<tr>
<td>reed</td>
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<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>bean</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>corn</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>Concept Types</td>
<td>nonalcoholic</td>
<td>hot</td>
<td>alcoholic</td>
<td>caffeinic</td>
<td>sparkling</td>
<td></td>
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<td>---------------</td>
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<td>-----------</td>
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<td></td>
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<tr>
<td>Tea</td>
<td>x</td>
<td>x</td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>Wine</td>
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<td>x</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Beer</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cola</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Left: Example specification;

Right: Corresponding context.

Attribute factored lattice form for the input specification of Figure 1.
Advanced FCA: nested diagrams.$^1$

Figure 5: A nominal scale for the world’s oldest trees

Figure 4: An interval scale for the world’s oldest trees

Figure 6: A nested line diagram combining figures 4 and 5
Advanced FCA: merging diagrams\textsuperscript{1}.
Def.: A **formal concept** is a pair \((A, B)\) where

- \(A\) is a set of objects (the **extent** of the concept),
- \(B\) is a set of attributes (the **intent** of the concept),
- \(A \times B\) is a maximal rectangle in the binary relation.

The extent (yellow rows) contain a common set of attributes.

### National Parks in California

<table>
<thead>
<tr>
<th></th>
<th>NPS Guided Tours</th>
<th>Hiking</th>
<th>Horseback Riding</th>
<th>Swimming</th>
<th>Boating</th>
<th>Fishing</th>
<th>Bicycle Trail</th>
<th>Cross Country Trail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cabrillo Natl. Mon.</td>
<td></td>
<td></td>
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<tr>
<td>Channel Islands Natl. Park</td>
<td></td>
<td></td>
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<tr>
<td>Death Valley Natl. Mon.</td>
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<td>Devils Postpile Natl. Mon.</td>
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<tr>
<td>Fort Point Natl. Historic Site</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>Golden Gate Natl. Recreation Area</td>
<td></td>
<td></td>
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<tr>
<td>John Muir Natl. Historic Site</td>
<td></td>
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<tr>
<td>Joshua Tree Natl. Mon.</td>
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<td>Kings Canyon Natl. Park</td>
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<td>Lassen Volcanic Natl. Park</td>
<td></td>
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<tr>
<td>Lava Beds Natl. Mon.</td>
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<tr>
<td>Muir Woods Natl. Mon.</td>
<td></td>
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<tr>
<td>Pinnacles Natl. Mon.</td>
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<tr>
<td>Point Reyes Natl. Seashore</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Redwood Natl. Park</td>
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<td></td>
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</tr>
<tr>
<td>Santa Monica Mts. Natl. Recr. Area</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Sequoia Natl. Park</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Whiskeytown-Shasta-Trinity Natl. Recr. Area</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>Yosemite Natl. Park</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>
The **blue** concept is a **subconcept** of the **yellow** one, since its extent is contained in the yellow one.

### National Parks in California

<table>
<thead>
<tr>
<th></th>
<th>NPS Guided Tours</th>
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<tbody>
<tr>
<td>Cabrillo Natl. Mon.</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Channel Islands Natl. Park</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Death Valley Natl. Mon.</td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Devils Postpile Natl. Mon.</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fort Point Natl. Historic Site</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Golden Gate Natl. Recreation Area</td>
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<td>X</td>
<td>X</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>John Muir Natl. Historic Site</td>
<td></td>
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<tr>
<td>Joshua Tree Natl. Mon.</td>
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<tr>
<td>Kings Canyon Natl. Park</td>
<td>X</td>
<td>X</td>
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<tr>
<td>Lassen Volcanic Natl. Park</td>
<td></td>
<td>X</td>
<td>X</td>
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<tr>
<td>Lava Beds Natl. Mon.</td>
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<tr>
<td>Muir Woods Natl. Mon.</td>
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<tr>
<td>Pinnacles Natl. Mon.</td>
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<tr>
<td>Point Reyes Natl. Seashore</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Redwood Natl. Park</td>
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<td>X</td>
<td>X</td>
<td>X</td>
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<tr>
<td>Santa Monica Mts. Natl. Recr. Area</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
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<tr>
<td>Sequoia Natl. Park</td>
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<tr>
<td>Whiskeytown-Shasta-Trinity Natl. Recr. Area</td>
<td></td>
<td>X</td>
<td>X</td>
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<tr>
<td>Yosemite Natl. Park</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
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</tbody>
</table>
The **concept lattice** of the National Parks in California:

- **Bicycle Trail**
- **Fishing**
- **Hiking**
- **NPS Guided Tours**
- **Muir Woods Pinnacles**
- **Horseback Riding**
- **Swimming**
- **Fort Point**
- **Joshua Tree**
- **Lava Beds**
- **Death Valley**
- **Channel Islands**
- **Kings Canyon Sequoia**
- **Boating**
- **Cross Country Ski Trail**
- **Lassen Volcanic**
- **Yosemite**
- **Golder Gate Point Reyes**
- **Devils Postpile Redwood**
- **Santa Monica Mountains Whiskeytown-Shasta-Trinity**

National Parks in California:

<table>
<thead>
<tr>
<th>National Parks in California</th>
<th>[ ]</th>
<th>[ ]</th>
<th>[ ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cabrillo Natl. Mon.</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Channel Islands Natl. Park</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Death Valley Natl. Mon.</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Devil Postpile Natl. Mon.</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Great Basin Natl. Mon.</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Golden Gate Recreational Ar</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Lake Tahoe Recreational Ar</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Lassen Volcanic Natl. Mon.</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>[ ]</td>
<td>[ ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
<tr>
<td>Muir Woods Natl. Mon.</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Point Reyes Natl. Sta.</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Redwood Natl. Park</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Sequoia Natl. Park</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

Legend:

- [ ]: Present
- [ ]: Absent
Top, intermediate, and bottom logical concepts.

\[
\top = \langle \text{all structures, true sentences} \rangle
\]

\[
c = \langle \text{extent}(c), \text{intent}(c) \rangle
\]

\[
\bot = \langle \text{no structures, all sentences} \rangle
\]
Non-distributive lattice

Assertion: it is the diagram of a non-distributive lattice. That it is a lattice is a routine verification. To prove that it is not distributive, verify that

\[ p \land (q \lor r) = p \land 1 = p \quad \text{and} \quad (p \land q) \lor (p \land r) = 0 \lor 0 = 0. \]
Concepts are maximal rectangles

<table>
<thead>
<tr>
<th></th>
<th>female</th>
<th>male</th>
<th>young</th>
<th>old</th>
</tr>
</thead>
<tbody>
<tr>
<td>mother</td>
<td>x</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>father</td>
<td></td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>son</td>
<td></td>
<td>x</td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>daughter</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>

By attribute

By object
({Father, Mother, Son, Daughter }, {})

({Father, Son}, {Male})
({Mother, Daughter}, {Female})
({Son, Daughter}, {Young})

({Father}, {Old, Male})
({Mother}, {Old, Female})
({Son}, {Male, Young})
({Daughter}, {Female, Young})

({}, {Female, Male, Old, Young})
Examples

• The subset relation (\(\subseteq\)) represents partial order (reflexive, anti-symmetric, transitive <). When reading, there is an implied upward orientation e.g. lower < upper. A point is drawn for each element of the poset, and line segments representing relations are drawn between these points according to the following two rules:

1. If \(x < y\) in the poset, then the point corresponding to \(x\) appears lower in the drawing than the point corresponding to \(y\).

2. The line segment between the points corresponding to any two elements \(x\) and \(y\) of the poset is included in the drawing iff \(x\) covers \(y\) or \(y\) covers \(x\).

• Implicit relations reflexive and transitive
• Explicit relation anti-symmetry.
Examples

• **Transitive:** Several examples
  \[\{1\} \subset \{1,2\} \subset \{1,2,3\}\]

• **Reflexive:** Several examples \(\{1\} \subseteq \{1\}\)

• **Cover:** A cover is the transitive reflexive reduction of a partial order. An element \(z\) (e.g. \(\{1,2\}\)) of a partially ordered set above \((X,\leq)\) covers another element \(x\) (e.g. \(\{1\}\) and \(\{2\}\)) provided that there exists no third element \(y\) in the poset for which \(x \leq y \leq z\). If we have \(x \leq y \leq z\), then \(z\) is called an upper cover of \(x\) and \(x\) a lower cover of \(z\).
Examples

• **Cover**: A cover is the transitive reduction of a partial order. An element \( z \) (e.g. \( \{1,2\} \)) of a partially ordered set above \( (X,\leq) \) covers another element \( x \) (e.g. \( \{1\} \) and \( \{2\} \)) provided that there exists no third element \( y \) in the poset for which \( x \leq y \leq z \). If we have \( x \leq y \leq z \), then \( z \) is called an *upper cover of* \( x \) and \( x \) a *lower cover of* \( z \).
Examples

• Proper subsets of exactly one set.
  • \( \{1, 2\} \subset \{1, 2, 3\} \)
  • \( \{1, 3\} \subset \{1, 2, 3\} \)
  • \( \{2, 3\} \subset \{1, 2, 3\} \)
Examples

- $H$ is a partial, it is reflexive, anti-symmetric, and transitive because of the divisibility relation.
- $H$ has two maximal elements 24, 18
- $H$ has one minimal element 1
- $H$ has one first element 1
- $H$ has no last element because it has two maximal elements neither of which is a last element (they are not comparable).
Examples

<table>
<thead>
<tr>
<th></th>
<th>Female</th>
<th>Male</th>
<th>Young</th>
<th>Old</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mother</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Father</td>
<td></td>
<td>×</td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Son</td>
<td>×</td>
<td></td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>Daughter</td>
<td>×</td>
<td></td>
<td></td>
<td>×</td>
</tr>
</tbody>
</table>

\[\{\text{Father, Mother, Son, Daughter}\}, \emptyset\]
Examples

FCA develops a mathematical and computable theory of concepts, which consist of objects and attributes.
FCA formally represent a Galois connection between ordered sets of objects and attributes.
A Galois connection is a relation between two partially ordered sets (posets). In the FCA case the posets are objects and attributes.
Sets of formal concepts can be visualized.
Logical inference can be used using a computer.
FCA develops a mathematical theory which can represent concepts, which consist of objects and attributes.
This mathematical theory can be visualized in an intuitive way.
Concept Analysis can be used to identify groupings of objects that possess common attributes.
Examples

FCA can be used as: general knowledge representation, ontology and ontology merging, in UML transition and transition reduction, Constructing classification & taxonomies, Data mining, Conceptual information systems, Information retrieval systems, Semantic Web Formally modelling object oriented class hierarchies
Examples

- Hasse diagram of the subset relation ($\subseteq$) on the power set of $\{1, 2, 3\}$ ($\mathcal{P}\{1, 2, 3\}$).
- Implicitly and explicitly relations, reading conventions.
- Identify a transitive relation, a reflexive relation, a cover, a set that is a proper subset of exactly one set.
Examples

• Hasse diagram can represent a partial order.

• Locating: single maximal element, a minimal element, a first element and a last element.
Examples

- Constructing a concept lattice from a table.
- Listing objects and attributes on a lattice

<table>
<thead>
<tr>
<th></th>
<th>Female</th>
<th>Male</th>
<th>Young</th>
<th>Old</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mother</td>
<td>×</td>
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<tr>
<td>Father</td>
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<tr>
<td>Son</td>
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<td></td>
<td>×</td>
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</tr>
<tr>
<td>Daughter</td>
<td>×</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>
Examples

• The advantages of using a concept lattice for knowledge representation?
• Application of formal concept analysis.
The END