Confidence Intervals and Hypothesis Test Sample Questions & Solutions

1. At the site of an archaeological dig two large burial sites have been found. In trying to decide whether the skeletons from these two sites come from different tribes anthropologists have measured the height of 40 skeletons from each burial site. The following are the sample statistics:

<table>
<thead>
<tr>
<th>Burial site 1</th>
<th>Mean = 163</th>
<th>Variance = 24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Burial site 2</td>
<td>Mean = 159</td>
<td>Variance = 12</td>
</tr>
</tbody>
</table>

(i) Calculate a 95% confidence interval for the population mean for skeletons from burial site 1.

\[ \bar{X} = 163 \quad S^2 = 24 \quad S = \sqrt{24} = 4.89 \]

95% CI = \[ \bar{X} \pm t_{\alpha/2,n-1} \times \frac{S}{\sqrt{n}} \]

\[ t_{0.05/2,39} = 2.021 \]

\[ = 163 \pm 2.021 \times \frac{4.89}{\sqrt{40}} \]

\[ (161.43, 164.56) \]

(ii) Calculate a 90% confidence interval for the mean difference between skeletons from each burial site.

\[ S_p = \sqrt{\frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}} \]

\[ = \sqrt{\frac{39 \times 24 + 39 \times 12}{78}} = \sqrt{18} = 4.24 \]

90% CI = \[ \bar{X}_1 - \bar{X}_2 \pm t_{\alpha/2,n_1+n_2-2} \times S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \]

\[ = (163 - 159) \pm 1.671 \times 4.24 \times \sqrt{\frac{1}{40} + \frac{1}{40}} \]

\[ (2.42, 5.58) \]
Conduct a hypothesis test, that the mean height of skeletons from each burial site is the same (use \( \alpha = 0.05 \)). Calculate the \( p \)-value for this test and state your conclusion clearly.

**Sol**

\( \bar{X}_1 = 163 \quad S^2_1 = 24 \quad n_1 = n_2 = 40 \)

\( \bar{X}_2 = 159 \quad S^2_2 = 12 \)

1. From the problem context, identify the parameter of interest.

\( \mu \)

2. State the null hypothesis, \( H_0 \).

\( H_0: \mu_1 = \mu_2 \)

3. Specify an appropriate alternative hypothesis, \( H_1 \).

\( H_1: \mu_1 \neq \mu_2 \)

4. Choose a significance level, \( \alpha \).

\( \alpha = 0.05 \)

5. Determine an appropriate test statistic.

\[
T_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad S_p = \sqrt{\frac{(n_1 - 1) S^2_1 + (n_2 - 1) S^2_2}{n_1 + n_2 - 2}}
\]

6. State the rejection region for the statistic.

\( T_0 < -2.021 \text{ or } T_0 > 2.021 \)

7. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute that value.

\[
S_p = \sqrt{\frac{(n_1 - 1) S^2_1 + (n_2 - 1) S^2_2}{n_1 + n_2 - 2}} = \sqrt{\frac{39 \times 24 + 39 \times 12}{78}} = \sqrt{18} = 4.24
\]

\[
T_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{163 - 159}{4.24 \sqrt{\frac{1}{40} + \frac{1}{40}}} = 4.22
\]

8. Because \( T_0 \) is in the rejection region we reject the null hypothesis that
there is no difference between the heights of the skeletons from the two burial sites and state that there is a difference.

2 An article in the Journal of Agricultural Science investigated means of wheat grain crude protein content (CP) and Hagburg falling number (HFN) surveyed in the UK. The analysis used a variety of nitrogen fertilizer applications (kg N/ha), temperature (°C), and total monthly rainfall (mm). The data shown below describe temperatures for wheat grown at Harper Adams Agricultural college between 1982 and 1993. The temperatures measured in June were obtained as follows:

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Temperature</th>
<th>Temperature</th>
<th>Temperature</th>
<th>Temperature</th>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.2</td>
<td>14.2</td>
<td>14.0</td>
<td>12.2</td>
<td>14.4</td>
<td>12.5</td>
</tr>
<tr>
<td>14.3</td>
<td>14.2</td>
<td>13.5</td>
<td>11.8</td>
<td>15.2</td>
<td></td>
</tr>
</tbody>
</table>

Assume that the standard deviation is known to be \( \alpha = 0.5 \).

(i) Construct a 99% two-sided confidence interval on the mean temperature.

\[
\bar{X} = 13.77 \quad S^2 = 1.32 \quad S = \sqrt{1.32} = 1.1
\]

99% CI = \( \bar{X} \pm t_{\alpha/2, n-1} \times \frac{S}{\sqrt{n}} \)

\[
t_{0.01/2, 10} = 3.169
\]

\[
= 13.77 \pm 3.169 \times \frac{1.1}{\sqrt{11}}
\]

(12.67, 14.87)

(ii) Construct a 95% lower-confidence bound on the mean temperature.

\[
\bar{X} = 13.77 \quad S^2 = 1.32 \quad S = \sqrt{1.32} = 1.1
\]

95% Lower Bound = \( \bar{X} - t_{\alpha, n-1} \times \frac{S}{\sqrt{n}} \)

\[
t_{0.05, 10} = 1.812
\]

\[
= 13.77 - 1.812 \times \frac{1.1}{\sqrt{11}}
\]

= 13.14

(iii) Suppose that we wanted to be 95% confident that the error in estimating the mean temperature is less than 2° C. What sample size should be used? (Sample sizing is not examinable)
A random sample has been taken from a normal distribution and the following confidence intervals have been constructed using the same data: (37.53, 49.87) and (35.59, 51.81).

(i) What is the value of the sample mean?

\[ \text{Sol} \quad \mu = \frac{\overline{x}}{\sigma} = \frac{2}{1.1} = 1.81 \Rightarrow n \sim 10 \]

(ii) One of these intervals is a 99% CI and the other is a 95% CI. Which one is the 95% CI? Why?

\[ \text{Sol} \quad 99\% \text{ CI} \Rightarrow (35.59, 51.81) \]
\[ \text{95}\% \text{ CI} \Rightarrow (37.53, 49.8) \]

The 95% CI is narrower than the 99% CI as the confidence level is lower.

Consider the computer output below:

<table>
<thead>
<tr>
<th>Variable</th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
<th>SE Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>15</td>
<td>8.69</td>
<td>?</td>
<td>0.443</td>
</tr>
</tbody>
</table>

(i) Calculate the missing value (indicated with a ?).

\[ \text{Sol} \quad SE \text{ Mean} = \frac{s}{\sqrt{n}} \Rightarrow s = SE \text{ Mean } \times \sqrt{n} = 0.0443 \times \sqrt{15} = 1.72 \]

(ii) Calculate a 95% confidence interval about the mean. What conclusions would you draw from this result?

\[ \text{Sol} \quad \bar{X} = 8.69 \quad S = 1.72 \]

\[ 95\% \text{ CI} = \bar{X} \pm t_{a/2,n-1} \times \frac{S}{\sqrt{n}} \]
\[ t_{0.05/2,14} = 2.145 \]
\[ = 8.69 \pm 2.145 \times \frac{1.72}{\sqrt{15}} \]
\[ = (7.74, 9.64) \]

(iii) The example above is taken from an experiment done by a freight company that wanted to test the claim that deliveries were completed within 8 working days.
being sent. What conclusions can be drawn about delivery times?

**Sol** As the 8 lies within the 95% confidence interval this implies that the company is correct in claiming that orders will be delivered within 8 days 95% of the time.

5 A manufacturing company is testing the use of two different alloys in the formulation of solder. Tests of the melting points (in °C) of solder formulations are undertaken using each different alloy. 50 tests are performed using each of the alloys and the melting points are measured. The following are the sample statistics:

| Sample 1: [Alloy A] | Mean = 236 | Variance = 25 |
| Sample 2: [Alloy B] | Mean = 224 | Variance = 15 |

(i) Calculate a 90% confidence interval for the population mean for Alloy A.

**Sol**

\[
\bar{X} = 236 \quad S^2 = 25 \quad S = \sqrt{25} = 5
\]

95% CI = \( \bar{X} \pm t_{\alpha/2,n-1} \times \frac{S}{\sqrt{n}} \)

\( t_{0.1/2.49} = 1.684 \)

\[
= 236 \pm 1.684 \times \frac{5}{\sqrt{50}}
\]

(234.80, 237.19)

(ii) Calculate a 95% confidence interval for the mean difference between solder formulations using the two alloys.
Conduct an hypothesis test, that the melting points of the solder formulation using alloy A is greater than he melting point for the solder formulation using alloy B (use $\alpha = 0.025$). Calculate the $p$-value for this test and state your conclusion clearly.

**Sol**

1. From the problem context, identify the parameter of interest.
   \[ \mu_1 - \mu_2 = 0 \]
2. State the null hypothesis, $H_0$.
   \[ H_0: \mu_1 - \mu_2 = 0 \]
3. Specify an appropriate alternative hypothesis, $H_1$.
   \[ H_1: \mu_1 - \mu_2 > 0 \]
4. Choose a significance level, $\alpha$.
   \[ \alpha = 0.05 \]
5. Determine an appropriate test statistic.
   \[
   T_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \\
   S_p = \sqrt{\frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}}
   \]
6. State the rejection region for the statistic.

\[
\sqrt{49 \times 5 + 49 \times 3.87 \over 98} = 2.11
\]

95% CI = $\bar{X}_1 - \bar{X}_2 \pm t_{\alpha/2,n_1+n_2-2} \times S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

\[
= (236 - 224) \pm 2 \times 2.11 \times \sqrt{\frac{1}{50} + \frac{1}{50}}
\]

(11.156,12.844)

(iii) Conduct an hypothesis test, that the melting points of the solder formulation using alloy A is greater than the melting point for the solder formulation using alloy B (use $\alpha=0.025$). Calculate the $p$-value for this test and state your conclusion clearly.

**Sol**

$\bar{X}_1 = 236 \quad S_1^2 = 25 \quad n_1 = n_2 = 50$

$\bar{X}_2 = 224 \quad S_2^2 = 15$

1. From the problem context, identify the parameter of interest.
   \[ \mu_1 - \mu_2 = 0 \]
2. State the null hypothesis, $H_0$.
   \[ H_0: \mu_1 - \mu_2 = 0 \]
3. Specify an appropriate alternative hypothesis, $H_1$.
   \[ H_1: \mu_1 - \mu_2 > 0 \]
4. Choose a significance level, $\alpha$.
   \[ \alpha = 0.05 \]
5. Determine an appropriate test statistic.
   \[
   T_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \\
   S_p = \sqrt{\frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}}
   \]
6. State the rejection region for the statistic.
\[ T_0 > 2 \]

7. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute that value.

\[
S_p = \sqrt{\frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}}
\]

\[
= \sqrt{\frac{49 \times 5 + 49 \times 3.87}{98}} = 2.11
\]

\[
T_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{236 - 224}{2.11 \sqrt{\frac{1}{50} + \frac{1}{50}}} = 28.44
\]

8. Decide whether or not \( H_0 \) should be rejected and report that in the problem context. As 22.22 is greater than 2 and lies within the rejection region the \( H_0 \) is rejected at a confidence level of 97.5%. This means that the mean melting point of solder using Alloy A is greater than the melting point of the solder using Alloy B.

6 A polymer is manufactured in a batch chemical process. Viscosity measurements are normally made on each batch, and long experience with the process has indicated that the variability in the process is fairly stable with \( \sigma = 20 \).

Fifteen batch viscosity measurements are given as follows:

724 718 776 760 745 759 795
756 742 740 761 749 739 747
742

A process change is made which involves switching the type of catalyst used in the process. Following the process change, eight batch viscosity measurements are taken:

735 775 729 755 783 760 738 780

Assume the process variability is unaffected by the catalyst change. If the difference in the mean batch viscosity is 10 or less the manufacturer would like to detect it with a high probability.
Formulate and test an appropriate hypothesis using $\alpha = 0.10$. What are your conclusions? Find the $p$-value.

Sol

1) The parameter of interest is the difference in mean batch viscosity before and after the process change, $\mu_1 - \mu_2$

2) $H_0 : \mu_1 - \mu_2 = 10$

3) $H_1 : \mu_1 - \mu_2 < 10$

4) $\alpha = 0.10$

5) The test statistic is

$$z_0 = \frac{(\bar{x}_1 - \bar{x}_2) - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

6) Reject $H_0$ if $z_0 < -z_{0.1}$ where $z_{0.1} = -1.28$

7) $\bar{x}_1 = 750.2$  $\bar{x}_2 = 756.88$  $\Delta_0 = 10$

$\sigma_1 = 20$  $\sigma_2 = 20$

$n_1 = 15$  $n_2 = 8$

$$z_0 = \frac{(750.2 - 756.88) - 10}{\sqrt{\frac{(20)^2}{15} + \frac{(20)^2}{8}}} = -1.90$$

8) Because $-1.90 < -1.28$ reject the null hypothesis and conclude the process change has increased the mean by less than 10.

$$P\text{-value} = P(Z \leq -1.90) = 1 - P(Z \leq 1.90) = 1 - 0.97128 = 0.02872$$

Find a 90% confidence interval on the difference in mean batch viscosity resulting from the process change.

Sol

Case 1: Before Process Change

- $\mu_1$ = mean batch viscosity before change change
- $\bar{x}_1 = 750.2$
- $\sigma_1 = 20$
- $n_1 = 15$

Case 2: After Process Change

- $\mu_2$ = mean batch viscosity after change
- $\bar{x}_2 = 756.88$
- $\sigma_2 = 20$
- $n_2 = 8$

90% confidence on $\mu_1 - \mu_2$, the difference in mean batch viscosity before and after process change:
(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}

(750.2 - 756.88) - 1.645 \sqrt{\frac{(20)^2}{15} + \frac{(20)^2}{8}} \leq \mu_1 - \mu_2 \leq (750.2 - 756.88) + 1.645 \sqrt{\frac{(20)^2}{15} + \frac{(20)^2}{8}}

-21.08 \leq \mu_1 - \mu_2 \leq 7.72

We are 90% confident that the difference in mean batch viscosity before and after the process change lies within –21.08 and 7.72. Because 0 is contained in this interval we can conclude with 90% confidence that the mean batch viscosity was unaffected by the process change.

(iii) Compare the results of parts (i) and (ii) and discuss your findings.

Sol

Parts (a) and (b) above give evidence that the mean batch viscosity change is less than 10. This conclusion is also seen by the confidence interval given in a previous problem because the interval does not contain the value 10. The upper endpoint of the confidence interval is only 7.72.

7 Two machines are used for filling plastic bottles with a net volume of 16.10 mL. The fill volume can be assumed normal, with a standard deviation of $\sigma_1 = 0.020$ mL and $\sigma_2 = 0.025$ mL for machines 1 and 2 respectively. A member of the quality engineering staff suspects that both machines fill to the same mean net volume, whether or not this volume is 16.10 mL. A random sample of 10 bottles is taken from the output of each machine.

<table>
<thead>
<tr>
<th>Machine 1</th>
<th>Machine 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.017</td>
<td>16.009</td>
</tr>
<tr>
<td>16.014</td>
<td>16.006</td>
</tr>
<tr>
<td>16.012</td>
<td>16.004</td>
</tr>
<tr>
<td>16.015</td>
<td>16.007</td>
</tr>
<tr>
<td>16.013</td>
<td>16.005</td>
</tr>
</tbody>
</table>

Do you think that the engineer is correct? Use $\alpha = 0.05$. What is the $p$-value for this test?

1) The parameter of interest is the difference in fill volume $\mu_1 - \mu_2$. Note that $\Delta_0 = 0$.

2) $H_0 : \mu_1 - \mu_2 = 0$ or $\mu_1 = \mu_2$

3) $H_1 : \mu_1 - \mu_2 \neq 0$ or $\mu_1 \neq \mu_2$

4) $\alpha = 0.05$

5) The test statistic is
\[ z_0 = \frac{(\bar{x}_1 - \bar{x}_2) - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \]

6) Reject H_0 if \( z_0 < -z_{\alpha/2} = -1.96 \) or \( z_0 > z_{\alpha/2} = 1.96 \)

7) \( \bar{x}_1 = 16.014 \quad \bar{x}_2 = 16.006 \)
\( \sigma_1 = 0.020 \quad \sigma_2 = 0.025 \)
\( n_1 = 10 \quad n_2 = 10 \)

\[
z_0 = \frac{(16.014 - 16.006)}{\sqrt{\frac{(0.020)^2}{10} + \frac{(0.025)^2}{10}}} = 0.79
\]

8) Because \(-1.96 < 0.79 < 1.96\), do not reject the null hypothesis. There is not sufficient evidence to conclude that the two machine fill volumes differ at \( \alpha = 0.05 \).

(Not Examinable) \( P\)-value = \( 2(1 - \Phi(0.79)) = 2(1 - 0.7852) = 0.429 \)

(ii) Calculate a 95% confidence interval on the difference in means. Provide a practical interpretation of this interval.

Sol
\[
(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}
\]

\[
(16.014 - 16.006) - 1.96 \sqrt{\frac{(0.02)^2}{10} + \frac{(0.025)^2}{10}} \leq \mu_1 - \mu_2 \leq (16.014 - 16.006) + 1.96 \sqrt{\frac{(0.02)^2}{10} + \frac{(0.025)^2}{10}}
\]

\[-0.0168 \leq \mu_1 - \mu_2 \leq 0.0328
\]

With 95% confidence, we believe the true difference in the mean fill volumes is between \(-0.0098\) and \(0.0298\). Because 0 is contained in this interval, we can conclude there is no significant difference between the means.